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A new construction for skew multivariate distributions

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Abstract

This paper considers a new approach to develop a very general class of skew multivariate distributions. The approach is based on a linear combination of an elliptically distributed random variable with a linear constraint. Using this approach two different classes of multivariate distributions are constructed based on original distribution. These new classes include different types of skew normal (type A and type B) and other skew elliptical distributions, exist in the literature. We also derive the moment generating function, marginal and conditional density of our proposed classes of distributions. Straightforward explanations are applied to demonstrate the relationships among previous approaches by others with our proposed class of skew distributions.

AMS 1991 subject classification: 60E05; 62H10

Keywords: Elliptical distribution; Linear combination; Linear constraint; Moment generating function; Multivariate normal; Skewness

1. Introduction

In many fields such as economics, psychology and sociology, sometimes error structures in a regression type models no longer satisfy symmetric property. Often there is a presence of high skewness. To preserve important properties it is natural to decompose some distributions into original symmetric portion and accumulated linearly constrained portion to

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demonstrate the prevalence of skewness. The first appearance of the skew normal distribution (SN, hereafter) dates back to Roberts [16], who considered $Z = \min(X, Y)$, where (X, Y) is correlated bivariate normal random variable. After that several papers have been written on skew distributions, both on univariate and multivariate set up. They include Azzalini and Dalla Valle [5], who proposed Type A-MSN (multivariate skew normal) (α, μ, Ω) with density of the form $f(x; \alpha, \mu, \Omega) = 2\phi_p(x; \mu, \Omega)\Phi(\alpha'(x-\mu))$, $x \in \Re^p$. Recently Azzalini and Capitanio [4], Gupta et al. [11] and Domínguez-Molina et al. [8] also studied multivariate skew normal distributions. Gupta et al. [11] proposed Type B-MSN (multivariate skew normal) (D, μ, Ω) with density of the form $f(x; D, \mu, \Omega) = \frac{\Phi_p(D(x-\mu);0,I)}{\Phi_p(D(0;0,I+D\Omega D')}\ \phi_p(x; \mu, \Omega)$, $x \in \Re^p$. The paper of Domínguez-Molina et al. [8] includes Gupta et al. [11] as a special case. Recently Branco and Dey [7], and Sahu et al. [17] constructed two new classes of multivariate skew elliptical distributions. They used a conditional method by introducing positive random vector components. Whereas Branco and Dey [7] applied similar idea of Azzalini and Dalla Valle [5] to construct multivariate skew elliptical distribution. Liseo and Loperfido [14] constructed a broad class of multivariate skew normal distributions by location vector mixture within linear constraints. Gupta et al. [11] and Sahu et al. [17] method is a special case of that of Liseo and Loperfido [14]. However, Domínguez-Molina et al. [8] is not a special case of that of Liseo and Loperfido [14]. Jones [13] applied the marginal replacement method in constructing skew multivariate distributions with application to skewing spherically symmetric distributions. Jones' method is a quite self-contained and flexible way to construct skew distributions. To preserve some properties of the original symmetric distributions, Jones concentrated on two-dimensional skew construction and proposed only one component replacement for multivariate skew distributions. Skewness construction by "truncation" was discussed in Arnold et al [1], subsequently Arnold and Beaver [2] gave a comprehensive review over the literatures on multivariate skewness construction, interpretation and characterizing property. Their explanations for skewness mechanism is related to hidden truncation and/or selective reporting.

Considering a p-dimensional random vector W which has an elliptically contoured distribution, e.g.,

$$W = \mu + \Sigma^{1/2} Z, \tag{1}$$

where Z is a spherically symmetric random vector, μ is a location vector and Σ is a scale matrix. Such a random vector Z may be represented as $Z = R(U_1, \ldots, U_p)$, where R and u are independent random variables, with $P(R > 0) = 1$ and $R \sim F_R$, R is uniformly distributed over the unit p-sphere. The distribution of Z determines the distribution of W. Consequently the components of W will be dependent, so are the components of Z. It is known that the components of Z will be independent only if R has a particular chi-distribution, i.e., R is a constant multiple of the square root of a χ^2 distribution with p degrees of freedom. Then Z will be $N(0, I)$ and W will also have independent components if $\Sigma^{1/2}$ is orthogonal. Starting with a (p+1)-dimensional elliptically contoured random vector of the form (1), say $(W_0, \ldots, W_p)$, Arnold and Beaver [2] considered conditional distribution of $(W_1, \ldots, W_p)$ given that $W_0 > c$, which they called a p-dimensional skew-elliptical density. If we consider a special case in which Z has an appropriate chi-distribution, this skewed model will reduce to the model obtained by applying transformations to the density

$$f(w; 0, \lambda_1) = 2\left[\prod_{i=1}^{p} \psi(w_i)\right] \Psi(\lambda_1' w),$$

where ψ is the pdf, Ψ is the cdf. They chose Z such that U has Cauchy marginal to get skewed-Cauchy distribution. Suppose $W_1, \ldots, W_p$ and U are independent random variables with densities given by $\psi_1(w_1), \ldots, \psi_p(w_p)$ and cdf $\Psi_0(u)$. The conditional distribution of W given that $\lambda_0 + \lambda_1' W > U$, where $\lambda_0 \in \Re$ and $\lambda_1 \in \Re^p$ is

$$f_{W|A}(w) = \frac{\prod_{i=1}^{p} \psi(w_i) \Psi_0(\lambda_0 + \lambda_1' w)}{P(A)},$$

where $A = \{\lambda_0 + \lambda_1 W > U\}$. If we assume a joint density of W as $\psi(w)$, the above density will be

$$f_{W|A}(w) = \frac{\psi(w) \Psi_0(\lambda_0 + \lambda_1' w)}{P(A)}.$$

Arnold and Beaver [2] point out that, if we begin with $\psi(w)$ an elliptically contoured density then this preceding formula included the Branco and Dey [7] approaches as special cases. A different type of skew-elliptical distribution by Azzalini and Capitanio [4] is also a special case of it.

In this paper, we propose a new approach based on linear constraints and linear combination (LCLC) to develop a new class of skew-elliptical distributions. This class includes Liseo and Loperfido [14] and Domínguez-Molina et al. [8] density in a more general sense. We also derive the corresponding density function for each case of LCLC skewness constructions. Liseo and Loperfido [14] claim Type B-MSN density does not generalize Type A-MSN density. We illustrate the relationship between Type A-MSN and Type B-MSN and point out that all of the aforementioned skewness constructions are embedded within our scheme although ours is a dichotomous construction.

The format of the paper is as follows. Section 2 develops our main results on a more general constructions for multivariate skew distribution. Section 3 summarizes all of the special cases of our construction. They include previous authors' results on multivariate skew density functions to date. Section 4 develops the moment generating function, marginal density and conditional density and some properties of our multivariate skew distribution, especially multivariate skew normal distribution. Section 5 provides some concluding remarks.

2. Main results

2.1. Elliptical distributions

Consider a p-dimensional random vector X having probability density function (pdf) of the form

$$f(x|\mu, \Omega; g^{(p)}) = |\Omega|^{-\frac{1}{2}} g^{(p)}((x-\mu)'\Omega^{-1}(x-\mu)), x \in \Re^p,$$

where $g^{(p)}(u)$ is a non-increasing function from R^+ to R^+ defined by

$$g^{(p)}(u) = \frac{\Gamma(p/2)}{\pi^{p/2}} \frac{g(u;\, p)}{\int_0^\infty r^{p/2-1} g(r;\, p)\, dr}$$

and $g(u;\, p)$ is a non-increasing function from R^+ to R^+ such that the integral $\int_0^\infty r^{p/2-1} g(r;\, p)\, dr$ exists. In this paper, we will always assume the existence of the pdf $f(x|\mu, \Omega;\, g^{(p)})$. The function $g^{(p)}$ is often called the *density generator* of the random vector *X*. Note that the function $g(u;\, p)$ provides the kernel of *X* and other terms in $g^{(p)}$ constitute the normalizing constant for the density *f*. In addition the function *g*, hence $g^{(p)}$, may depend on other parameters which would be clear from the context. For example, in case of *t* distributions the additional parameter will be the degrees of freedom. The density *f* defined above represents a broad class of distributions called the *elliptically symmetric distribution* and we will use the notation

$$X \sim El(\theta, \Omega;\, g^{(p)}),$$

henceforth in this article. Let $F(x|\theta, \Omega;\, g^{(p)})$ denote the cumulative density function (cdf) of *X* where $X \sim El(\theta, \Omega;\, g^{(p)})$.

We consider two examples, namely the multivariate normal and *t* distributions.

Example 1 (*Multivariate normal*). Let $g(u;\, p) = \exp(-u/2)$. Then straightforward calculation yields

$$g^{(p)}(u) = \frac{e^{-u/2}}{(2\pi)^{p/2}}.$$

Then

$$f(x|\mu, \Omega;\, g^{(p)}) = \frac{1}{(2\pi)^{p/2}} |\Omega|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)'\Omega^{-1}(x-\mu)\right), x \in \Re^p,$$

which is the pdf of the *p*-variate normal distribution with mean vector θ and covariance matrix Ω. We denote this distribution by $N_p(\theta, \Omega)$ and the pdf by $N_p(x|\theta, \Omega)$ henceforth.

Example 2 (*Multivariate t*). Let

$$g(u;\, p, v) = \left(1 + \frac{u}{v}\right)^{-(v+p)/2}, v > 0.$$

Here *g* depends on the additional parameter *v*, the degrees of freedom. Then straightforward calculation yields

$$g^{(p)}(u;\, v) = \frac{\Gamma(\frac{v+p}{2})}{\Gamma(\frac{v}{2})(v\pi)^{p/2}} g(u;\, p, v).$$

Hence

$$f(x|\mu, \Omega;\, g^{(p)}) = \frac{\Gamma(\frac{v+p}{2})}{\Gamma(\frac{v}{2})(v\pi)^{p/2}} |\Omega|^{-\frac{1}{2}}$$

$$\times\left(1+\frac{(x-\mu)'\Omega^{-1}(x-\mu)}{v}\right)^{-(v+p)/2}, x \in \Re^p,$$

which is the density of the p-variate t distribution with parameters θ, Ω and degrees of freedom v. We denote this distribution by $t_{p,v}(\theta, \Omega)$ and the density by $t_{p,v}(x|\theta, \Omega)$ henceforth. The subscript p will be omitted when it is equal to 1.

The following lemma will be useful for the rest of the paper.

Lemma 2.1. *If $X \sim El(\mu, \Omega; g^{(p)})$, and X is partitioned as $X = (X_1, X_2)'$, where X_1 is $p_1 \times 1$ and X_2 is $p_2 \times 1$ with*

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim El\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}; g^{(p)}\right), p = p_1 + p_2$$

then

$$X^{(1)}|X^{(2)} = x^{(2)} \sim El(\mu_{1.2}, \Omega_{11.2}; g^{(p_1)}_{q(x^{(2)})}),$$

where

$$\begin{cases} \mu_{1.2} &= \mu^{(1)} + \Omega_{12}\Omega_{22}^{-1}(x^{(2)} - \mu^{(2)}), \\ \Omega_{11.2} &= \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}, \\ q(x^{(2)}) &= (x^{(2)} - \theta^{(2)})'\Omega_{22}^{-1}(x^{(2)} - \theta^{(2)}), \\ g_a^{(p_1)}(u) &= \frac{\Gamma(p_1/2)}{\pi^{p/2}} \frac{g(a+u; p_1)}{\int_0^\infty r^{p_1/2-1} g(a+r; p_1)\, dr}, \\ g_a^{(1)}(u) &= \frac{g^{(p+1)}(u+a)}{g^{(p)}(a)}. \end{cases}$$

Proof. The proof follows from Fang et al. [12]. □

Lemma 2.2. *Suppose $X \sim El(\mu, \Omega, g^{(p)})$, and R is a matrix of order $k \times p$, then $RX \sim El(R\mu, R\Omega R', g^{(k)})$.*

2.2. General skew multivariate elliptical distribution

Suppose the random vector $Y \sim El(\theta, \Omega; g^{(p)})$ and satisfies the following linear constraint:

$$RY + d \leqslant 0, \tag{2}$$

where R is a given matrix of dimension $k \times p$, $k \leqslant p$ and d is a vector of dimension k.

Then defining p_c as the probability of the constraint set, we have

$$p_c = P(RY + d \leqslant 0) = P(RY - R\mu \leqslant -d - R\mu) = F(-d - R\mu|0, R\Omega R', g^{(k)}),$$

where F is the cdf of an elliptical distribution with location 0_k, scale $R\Omega R'$, density generator $g^{(k)}$, and 0_k is a k-dimensional vector of 0.

Further partitioning R as $R = (R_1, R_2)$, we can express constraint (2) as

$$R_1 Y_1 + R_2 Y_2 + d \leqslant 0, \tag{3}$$

where the dimension of the vectors of Y_1 and Y_2 are, respectively, p_1 and p_2. The dimension of R_1 is $k \times p_1$, the dimension of R_2 is $k \times p_2$. Here R_1 is of full row rank, and R_2 is an arbitrary matrix such that the multiplication matrix is valid.

2.2.1. Linear constraint and linear combination of type-1 (LCLC1)

Suppose the dimension of Y_1 is not equal to the dimension of Y_2, i.e., $p_1 \neq p_2$. Consider the distribution of $C_2 Y_2$, where C_2 is a non-singular square matrix with dimension $p_2 \times p_2$. Here Y_1 is used only as an auxiliary variable for producing skewness for Y_2, it does not show up in the final form of the skew distribution.

Theorem 2.2.1. *Under the linear conditions in LCLC1, given a non-singular $p_2 \times p_2$ matrix C_2, the density function for $X = C_2 Y_2$ under constraint (3) is*

$$\frac{F(-d - R_1(\mu_1 - \Omega_{12}\Omega_{22}^{-1}\mu_2) - (R_2 + R_1\Omega_{12}\Omega_{22}^{-1})C_2^{-1}x|0, \Omega_x, g_x^{(k)})}{F(-d - R\mu|0, R\Omega R', g^{(k)})} \times f(x|C_2\mu_2, C_2\Omega_{22}C_2', g^{(p_2)}), \tag{4}$$

where k is the number of rows in R,

$$\begin{cases} \Omega_x = & R_1(\Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21})R_1' \\ g_x^{(k)} = & g^{(k)}_{(C_2^{-1}x-\mu_2)'\Omega_{22}^{-1}(C_2^{-1}x-\mu_2)}. \end{cases}$$

Proof. Under linear constraint (3), the joint density function of (Y_1, Y_2) is

$$f_c(y_1, y_2) = \frac{f(y_1, y_2|\mu, \Omega; g^{(p)})}{F(-d - R\mu|0, R\Omega R'; g^{(k)})} \text{ on } R_1Y_1 + R_2Y_2 + d \leqslant 0.$$

Integrating out Y_1, we get the density of Y_2 as

$$\begin{aligned} f_c(y_2) &= \frac{1}{F(-d - R\mu|0, R\Omega R'; g^{(k)})} \int_{R_1Y_1 \leqslant -d - R_2Y_2} f(y_1, y_2|\mu, \Omega; g^{(p)})\, dy_1 \\ &= \frac{1}{F(-d - R\mu|0, R\Omega R'; g^{(k)})} \int_{R_1Y_1 \leqslant -d - R_2Y_2} \frac{f(y_1, y_2|\mu, \Omega; g^{(p)})}{f(y_2|\mu_2, \Omega_{22}; g^{(p_2)})} \\ &\quad \times f(y_2|\mu_2, \Omega_{22}; g^{(p_2)})\, dy_1 \\ &= \frac{1}{F(-d - R\mu|0, R\Omega R'; g^{(k)})} f(y_2|\mu_2, \Omega_{22}; g^{(p_2)}) \\ &\quad \times \int_{R_1Y_1 \leqslant -d - R_2Y_2} f(y_1|y_2, \mu, \Omega; g^{(p)})\, dy_1. \end{aligned}$$

Now using Lemma 2.1,

$$Y_1|Y_2 \sim El(\mu_{1.2}, \Omega_{11.2}; g^{(p_1)}_{q(Y_2)}),$$

where p_1 is the dimension of Y_1. Now using Lemma 2.2, we have

$$R_1Y_1|Y_2 \sim El(R_1\mu_{1.2}, R_1\Omega_{11.2}R_1'; g^{(k)}_{q(Y_2)}).$$

Thus

$$\begin{aligned} f_c(y_2) &= \frac{1}{F(-d - R\mu|0, R\Omega R'; g^{(k)})} f(y_2|\mu_2, \Omega_{22}; g^{(p_2)}) \\ &\quad \times \int_{R_1Y_1 \leqslant -d-R_2Y_2} f(y_1|y_2, \mu, \Omega; g^{(p)})\, dy_1 \\ &= \frac{1}{F(-d - R\mu|0, R\Omega R'; g^{(k)})} f(y_2|\mu_2, \Omega_{22}; g^{(p_2)}) \\ &\quad \times F(-d - R_2y_2|R_1(\mu_1 + \Omega_{12}\Omega_{22}^{-1}(y_2 - \mu_2)), R_1(\Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21})R_1', \\ &\quad g^{(k)}_{(y_2-\mu_2)'\Omega_{22}^{-1}(y_2-\mu_2)}). \end{aligned}$$

The final result follows after some simplifications. □

2.2.2. Linear constraint and linear combination of type-2 (LCLC2)

If the dimension of Y_1 is same as that of Y_2, i.e., $p_1 = p_2$, we consider the distribution of $C_1Y_1 + C_2Y_2$ under the constraint (3), further $R_1C_1^{-1} - R_2C_2^{-1}$ is full row rank, where C_1, C_2 are non-singular square matrices.

Theorem 2.2.2. *Under the conditions in LCLC2, the pdf for $X = C_1Y_1 + C_2Y_2$ is*

$$\frac{F(-d - C_3(\mu_1^{wx} - \Omega_{12}^{wx}\Omega_{22}^{wx\,-1}\mu_2^{wx}) - (R_2C_2^{-1} + C_3\Omega_{12}^{wx}\Omega_{22}^{wx\,-1})x|0, C_3\Omega_{11}^*C_3'; g^{(k)}_{a^*})}{F(-d - R\mu|0, R\Omega R'; g^{(k)})} \\ \times f(x|\mu_2^{wx}, \Omega_{22}^{wx}; g^{(p_1)}), \tag{5}$$

where

$$\begin{cases} \mu^{wx} &= \begin{pmatrix} \mu_1^{wx} \\ \mu_2^{wx} \end{pmatrix} = \begin{pmatrix} C_1\mu_1 \\ C_1\mu_1 + C_2\mu_2 \end{pmatrix}, \\ \Omega^{wx} &= \begin{pmatrix} \Omega_{11}^{wx} & \Omega_{12}^{wx} \\ \Omega_{21}^{wx} & \Omega_{22}^{wx} \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ C_1 & C_2 \end{pmatrix} \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \begin{pmatrix} C_1' & C_1' \\ 0' & C_2' \end{pmatrix}, \\ &= \begin{pmatrix} C_1\Omega_{11}C_1' & C_1\Omega_{11}C_1' + C_1\Omega_{12}C_2' \\ C_1\Omega_{11}C_1' + C_2\Omega_{21}C_1' & C_1\Omega_{11}C_1' + C_2\Omega_{21}C_1' + C_1\Omega_{12}C_2' + C_2\Omega_{22}C_2' \end{pmatrix}, \\ C_3 = & R_1C_1^{-1} - R_2C_2^{-1}, \\ \mu_1^* = & \mu_1^{wx} + \Omega_{12}^{wx}\Omega_{22}^{wx\,-1}(x - \mu_2^{wx}), \\ \Omega_{11}^* = & \Omega_{11}^{wx} - \Omega_{12}^{wx}\Omega_{22}^{wx\,-1}\Omega_{21}^{wx}, \\ a^* = & (x - \mu_2^{wx})'\Omega_{22}^{wx\,-1}(x - \mu_2^{wx}). \end{cases}$$

Proof. Consider the transformation

$$\begin{cases} W &= C_1Y_1, \\ X &= C_1Y_1 + C_2Y_2, \end{cases} \iff \begin{cases} Y_1 &= C_1^{-1}W, \\ Y_2 &= C_2^{-1}(X - W). \end{cases}$$

Clearly this is a one-to-one mapping. Now it follows that

$$\begin{aligned}
f_X(x) &= \frac{1}{p_c}\int f(x,w)\,dw \\
&= \frac{1}{p_c}\int_{R_1Y_1+R_2Y_2\leqslant -d} f(y_1(x,w),y_2(x,w))\left|\frac{\partial(y_1,y_2)}{\partial(x,w)}\right| dw \\
&= \frac{1}{p_c}\int_{R_1C_1^{-1}W+R_2C_2^{-1}(X-W)\leqslant -d} f(x,w)\,dw \\
&= \frac{1}{p_c}\int_{(R_1C_1^{-1}-R_2C_2^{-1})W\leqslant -d-R_2C_2^{-1}X} f(x,w)\,dw \\
&= \frac{1}{p_c}\int_{(R_1C_1^{-1}-R_2C_2^{-1})W\leqslant -d-R_2C_2^{-1}X} f(x)f(w|x)\,dw \\
&= \frac{1}{p_c}f(x)\int_{C_3W\leqslant -d-R_2C_2^{-1}X} f(w|x)\,dw.
\end{aligned}$$

Further we know

$$\begin{pmatrix} W \\ X \end{pmatrix} = \begin{pmatrix} C_1Y_1 \\ C_1Y_1+C_2Y_2 \end{pmatrix} \sim El\left(\begin{pmatrix} \mu_1^{wx} \\ \mu_2^{wx} \end{pmatrix}, \Omega^{wx}; g^{(p)}\right).$$

Now recall the properties of multivariate elliptical distribution. From Lemmas 2.1 and 2.2, it follows that

$$\begin{cases} W|X & \sim El(\mu_1^*, \Omega_{11}^*; g_{a^*}^{(p_1)}), \\ C_3W|X & \sim El(C_3\mu_1^*, C_3\Omega_{11}^*C_3'; g_{a^*}^{(k)}), \\ X & \sim El(\mu_2^{wx}, \Omega_{22}^{wx}; g^{(p_1)}). \end{cases}$$

Thus we have

$$\begin{aligned}
f_X(x) &= \frac{1}{p_c} f(x|\mu_2^{wx}, \Omega_{22}^{wx}; g^{(p_1)})F(-d-R_2C_2^{-1}x|C_3\mu_1^*, C_3\Omega_{11}^*C_3'; g_{a^*}^{(k)}) \\
&= \frac{F(-d-R_2C_2^{-1}x|C_3\mu_1^*, C_3\Omega_{11}^*C_3'; g_{a^*}^{(k)})}{F(-d-R\mu|0, R\Omega R'; g^{(k)})} f(x|\mu_2^{wx}, \Omega_{22}^{wx}; g^{(p_1)}).
\end{aligned}$$

The final result follows after some simplifications. □

Remark 2.1. The following notations are introduced to define the corresponding dimensions: X_p, μ_p, $\Omega_{p\times p}$, $R_{k\times p}$, d_k, $C_{p_2\times p_2}$, $g^{(p)}$, $C_{1_{p_1\times p_1}}$, $C_{2_{p_2\times p_2}}$. These notations will be used throughout the rest of the paper. For simplicity we will not identify dimensions of all the variables.

(i) From the proof of Theorem 2.2.2, we can see that if $R_1C_1^{-1} - R_2C_2^{-1}$ has full row rank, then LCLC2 is a reparameterization of LCLC1. Although from the construction point of view, they are of different classes, they enjoy same distribution properties under mild conditions.

(ii) We will use notation $X \sim GSME1(\mu, \Omega, R, d, C, g^{(p)})$ (Theorem 2.2.1) to represent Type 1-GSME (Generalized Skew Multivariate Elliptical) distribution.
(iii) We will use notation $X \sim GSME2(\mu, \Omega, R, d, C_1, C_2, g^{(p)})$ (Theorem 2.2.2) to represent Type 2-GSME (Generalized Skew Multivariate Elliptical) distribution. In this case the vector dimension combination is (p_1, p_2), with $p_1 = p_2$.
(iv) In view of the reparameterization (equivalence) property of LCLC1 and LCLC2, it is always good practice to focus on the evolved distribution of Y_2 or $Y_1 + Y_2$ under an appropriately chosen original distribution plus linear constraint, e.g., the full-rank square matrix C in (i), C_1 and C_2 in (iii) are redundant during the skew construction procedure. To avoid overparametrization and non-identifiability in estimation procedures we suggest imposing $C = I$, and $C_1 = C_2 = I$.

Example 3. Let $X_p = (X'_{1(p_1)}, X'_{2(p_2)})'$ be LCLC1 or LCLC2 multivariate skew normal distribution with density

$$\frac{1}{\Phi_p(b|\mu^\Delta, \Omega^\Delta)}\Phi_k(a - Bx|\mu^\star, \Omega^\star)\phi_p(x|\mu, \Omega)$$

then the marginal density function for $X_{1(p_1)}$ is

$$\begin{aligned} f_{X_1}(x_1) = {} & \frac{1}{\Phi_p(b|\mu^\Delta, \Omega^\Delta)}\phi_{p_1}(x_1|\mu_1, \Omega_{11})\Phi_k(a + B_2(\Omega_{21}\Omega_{11}^{-1}\mu_1 - \mu_2) \\ & -\mu^\star - (B_1 + B_2\Omega_{21}\Omega_{11}^{-1})x_1; 0, B_2(\Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12})B_2' + \Omega^\star), \end{aligned}$$

where

$$B = (B_{1(k,p_1)}, B_{2(k,p_2)}), \mu_p = \begin{pmatrix} \mu_{1(p_1)} \\ \mu_{2(p_2)} \end{pmatrix}, \mu_k^\star = \begin{pmatrix} \mu^\star_{1(k_1)} \\ \mu^\star_{2(k_2)} \end{pmatrix},$$

$$(p = p_1 + p_2, k = k_1 + k_2),$$

$$\Omega = \begin{pmatrix} \Omega_{11(p_1,p_1)} & \Omega_{12(p_1,p_2)} \\ \Omega_{21(p_2,p_1)} & \Omega_{22(p_2,p_2)} \end{pmatrix}, \Omega^\star = \begin{pmatrix} \Omega^\star_{11(k_1,k_1)} & \Omega^\star_{12(k_1,k_2)} \\ \Omega^\star_{21(k_2,k_1)} & \Omega^\star_{22(k_2,k_2)} \end{pmatrix}.$$

Proof. It follows that

$$\begin{aligned} f_{X_1}(x_1) &= \int f(x_1, x_2)\,dx_2 = \int g(x_1)g(x_2|x_1)\,dx_2 \\ &= \frac{\phi_{p_1}(x_1; \mu_1, \Omega_{11})}{\Phi_p(b; \mu^\Delta, \Omega^\Delta)} \int \Phi_p(a - Bx; \mu^\star, \Omega^\star) \\ &\quad \times\phi_{p_2}(x_2; \mu_2 + \Omega_{21}\Omega_{11}^{-1}(x_1 - \mu_1), \Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12})\,dx_2 \\ &\quad \times\{\text{Recall } \phi_p(x_1, x_2; \mu, \Omega) = \phi_{p_1}(x_1; \mu_1, \Omega_{11}) \\ &\quad \times\phi_{p_2}(x_2; \mu_2 + \Omega_{21}\Omega_{11}^{-1}(x_1 - \mu_1), \Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12})\} \\ &= \frac{\phi_{p_1}(x_1; \mu_1, \Omega_{11})}{\Phi_p(b; \mu^\Delta, \Omega^\Delta)} \int \Phi_p(a - B_1x_1 - B_2x_2; \mu^\star, \Omega^\star) \\ &\quad \times\phi_{p_2}(x_2; \mu_2 + \Omega_{21}\Omega_{11}^{-1}(x_1 - \mu_1), \Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12})\,dx_2 \end{aligned}$$

$$\times\{\text{Recall } B=(B_1,B_2) \text{ and } x=(x_1',x_2')'\}$$
$$=\frac{\phi_{p_1}(x_1;\mu_1,\Omega_{11})}{\Phi_p(b;\mu^{\Delta},\Omega^{\Delta})}\Phi_k(-(B_1+B_2\Omega_{21}\Omega_{11}^{-1})x_1;\mu^{\star}-a$$
$$+B_2(\mu_2-\Omega_{21}\Omega_{11}^{-1}\mu_1),B_2(\Omega_{22}-\Omega_{21}\Omega_{11}^{-1}\Omega_{12})B_2'+\Omega^{\star}). \qquad \square$$

Remark 2.2. The formula is same for both LCLC1 and LCLC2 setups, although they are developed from different skew mechanisms.

2.3. Closure of marginal and conditional distribution in LCLC1

In this section, we study the closure properties of the marginal and conditional distributions under the linear constraint and linear combination of first type.

Theorem 2.3.1. *The marginal density is closed under LCLC1 setup.*

Proof. Suppose

$$\begin{pmatrix} Z \\ Y_1 \\ Y_2 \end{pmatrix} \sim El\left(\begin{pmatrix} \nu \\ \mu_1 \\ \mu_2 \end{pmatrix}, \Omega, g^{(p)}\right).$$

We express linear constraints as $R_1Z+R_2Y+d\leqslant 0$, where $Y=(Y_1,Y_2)'$ and consider random vector C_2Y.

Let

$$X=\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}=C_2Y=\begin{pmatrix} C_{2,1} \\ C_{2,2} \end{pmatrix}Y,$$

then

$$\begin{pmatrix} Z \\ X \end{pmatrix} \sim El\left(\begin{pmatrix} \nu \\ C_2\mu \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & C_2 \end{pmatrix}\Omega\begin{pmatrix} I & 0 \\ 0 & C_2' \end{pmatrix}, g^{(p)}\right).$$

Recall the relationship

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}=C_2Y,\ Y=C_2^{-1}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix},\ R_2Y=R_2C_2^{-1}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

Based on the dimensions of X_1 and X_2, let $R_3=(\,R_{3,1}\ \ R_{3,2}\,)=R_2C_2^{-1}$, then the linear constraint $R_1Z+R_2Y+d\leqslant 0$ is equivalent to

$$(\,R_1\ \ R_{3,1}\,)\begin{pmatrix} Z \\ X_1 \end{pmatrix}+R_{3,2}X_2+d\leqslant 0.$$

Thus the marginal distribution of X_2 is $SME(\mu, \Omega, R, d, C, g^{(p)})$, where the parameters are

$$\begin{cases} R_1 &= (R_1, R_{3,1}), \\ R_2 &= R_{3,2}, \\ R &= (R_1, R_2), \\ d &= d, \\ \mu &= (v', (C_{2,1}\mu)', (C_{2,2}\mu)')', \\ \Omega &= \begin{pmatrix} I & 0 \\ 0 & C_2 \end{pmatrix} \Omega \begin{pmatrix} I & 0 \\ 0 & C_2' \end{pmatrix}, \\ C &= I, \\ g &= g. \end{cases}$$

Theorem 2.3.2. *The conditional density is closed under LCLC1 setup.*

Proof. Suppose

$$\begin{pmatrix} Z \\ Y \end{pmatrix} = \begin{pmatrix} Z \\ Y_1 \\ Y_2 \end{pmatrix} \sim El\left(\begin{pmatrix} v \\ \mu_1 \\ \mu_2 \end{pmatrix}, \Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}, g^{(p)} \right).$$

We express linear constraint as $R_1 Z + R_2 Y + d \leqslant 0$ and consider random vector $C_2 Y$.

Let

$$\Omega_{22} = \Omega_{yy} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, C_2 y = \begin{pmatrix} C_{2,1} y \\ C_{2,2} y \end{pmatrix},$$

where the dimension of $C_{2,1}$ is $p_{21} \times p_2$, the dimension of $C_{2,2}$ is $p_{22} \times p_2$, such that $p_2 = p_{21} + p_{22}$. Then

$$\begin{aligned} \begin{pmatrix} Z \\ X_1 \\ X_2 \end{pmatrix} &= \begin{pmatrix} Z \\ C_{2,1} Y \\ C_{2,2} Y \end{pmatrix} \\ &\sim El\left(\begin{pmatrix} v \\ C_{2,1}\mu \\ C_{2,2}\mu \end{pmatrix}, \begin{pmatrix} \Omega_{11} & \Omega_{12} C_{2,1}' & \Omega_{12} C_{2,2}' \\ C_{2,1}\Omega_{21} & C_{2,1}\Omega_{22} C_{2,1}' & C_{2,1}\Omega_{22} C_{2,2}' \\ C_{2,2}\Omega_{21} & C_{2,2}\Omega_{22} C_{2,1}' & C_{2,2}\Omega_{22} C_{2,2}' \end{pmatrix}, g^{(p)} \right). \end{aligned}$$

Using the property of elliptical distribution we get

$$\begin{pmatrix} Z \\ X_1 \end{pmatrix} | X_2 = x_2 \sim El(\mu^\star, \Omega^\star, g^{(p_1 + p_{21})}),$$

where

$$\begin{cases} \mu^\star = \begin{pmatrix} v \\ C_{2,1}\mu \end{pmatrix} + \begin{pmatrix} \Omega_{12} C_{2,2}' \\ C_{2,1}\Omega_{22} C_{2,2}' \end{pmatrix} (C_{2,2}\Omega_{22} C_{2,2}')^{-1} (x_2 - C_{2,2}\mu), \\ \Omega^\star = \begin{pmatrix} \Omega_{11} & \Omega_{12} C_{2,1}' \\ C_{2,1}\Omega_{21} & C_{2,1}\Omega_{22} C_{2,1}' \end{pmatrix} \\ \qquad - \begin{pmatrix} \Omega_{12} C_{2,2}' \\ C_{2,1}\Omega_{22} C_{2,2}' \end{pmatrix} (C_{2,2}\Omega_{22} C_{2,2}')^{-1} \begin{pmatrix} C_{2,2}\Omega_{21} & C_{2,2}\Omega_{22} C_{2,1}' \end{pmatrix}. \end{cases}$$

Using same partition as in Theorem 2.3.1, suppose $R_3 = (\, R_{3,1} \;\; R_{3,2}\,) = R_2 C_2^{-1}$. Then the linear constraint $R_1 Z + R_2 Y + d \leqslant 0$ is equivalent to

$$(\, R_1 \;\; R_{3,1}\,) \begin{pmatrix} Z \\ X_1 \end{pmatrix} + R_{3,2} X_2 + d \leqslant 0.$$

Thus the conditional distribution of X_1 given X_2 is $SME(\mu, \Omega, R, d, C_1, C_2, g^{(p_1+p_{21})})$, where

$$\begin{cases} R_1 &= R_1, \\ R_2 &= R_{3,1}, \\ R &= (R_1, R_2), \\ d &= R_{3,2} a_2 + d, \\ C_2 &= I, \\ \mu &= \mu^\star, \\ \Omega &= \Omega^\star, \\ g &= g. \end{cases}$$

3. Special cases

In this section we consider several examples from the literature and show how they can be constructed as a special case from LCLC1 or LCLC2 approaches.

3.1. *Type A-MSN* (α, μ, Ω)

Azzalini and Dalla Valle [5] constructed multivariate skew normal density of the form $f(x; \alpha, \mu, \Omega) = 2\phi_p(x; \mu, \Omega)\Phi(\alpha'(x-\mu))$, $x \in \Re^p$. Their method is included in our LCLC1 setup by choosing

$$\begin{cases} R_1 &= -1, \\ R_2 &= 0_p', \\ R &= (R_1, R_2), \\ d &= 0, \\ C_2 &= I_p, \\ \mu &= (0, \mu')', \\ \Omega &= \begin{pmatrix} 1 & \delta_1 \; \dots \; \delta_p, \\ \delta_1 & \\ \vdots & \Omega \\ \delta_k & \end{pmatrix}, \\ g(\mu) &= \exp(-\mu/2). \end{cases}$$

3.2. *Type B-MSN* (D, μ, Ω)

Gupta et al. [11] constructed multivariate skew normal density of the form $f(x; D, \mu, \Omega) = \frac{\Phi_p(D(x-\mu);0,I)}{\Phi_p(D(0;0,I+D\Omega D')}\phi_p(x; \mu, \Omega)$, $x \in \Re^p$. Their method is included in our LCLC1 setup by choosing

$$\begin{cases} R_1 & = I_{q\times q}, \\ R_2 & = -D_{q\times p}, \\ R & = (I_{q\times q}, -D_{q\times p}), \\ d & = 0_q, \\ C_2 & = I_{p\times p}, \\ \mu & = ((D\mu)', \mu')', \\ \Omega & = \begin{pmatrix} I_{q\times q} & 0 \\ 0 & \Sigma_{p\times p} \end{pmatrix}, \\ g(\mu) & = \exp(-\mu/2). \end{cases}$$

3.3. *Liseo and Loperfido class of MSN*

Liseo and Loperfido [14] constructed a broader class of multivariate skew normal distribution which includes both Type A-MSN as well as Type B-MSN as special cases. The density of their skew normal distribution is

$$\frac{\Phi_k(0; C\Delta(\Sigma^{-1}x + \Omega^{-1}\mu) + d, C\Delta C')}{\Phi_k(0; C\mu + d, C\Omega C')}\phi_p(x, \mu, \Sigma + \Omega),$$

where $\Delta = \Sigma(\Sigma + \Omega)^{-1}\Omega$. It is a special case of LCLC2 with

$$\mu = \begin{pmatrix} \mu \\ 0 \end{pmatrix}, \Omega = \begin{pmatrix} \Omega & 0 \\ 0 & \Sigma \end{pmatrix}, R_1 = C, R_2 = 0, C_1 = I_{p\times p}, C_2 = I_{p\times p}.$$

Another way to illustrate Liseo and Loperfido [14] method is to consider

$$X|X_0 \sim N(X_0, \Sigma) \text{ and } X_0 \sim N(\mu, \Omega) \text{ with linear constraint } K'X_0 + d \leqslant 0,$$

where the dimension of K is $p \times m$, $m \leqslant p$, such that K is full rank. The random variable X can be regarded as the sum of two independent random variables, of which one is a $N_p(\mu, \Omega)$ distribution with linear constraint $K'X + d \leqslant 0$, the other is a $N_p(0, \Sigma)$ distribution.

Liseo and Loperfido [14] claim that when $k = 1$, their approach is a slight generalization of the Type A-MSN distribution, whereas when $k = p$, their approach produces a representation of the Type B-MSN density. Here we observe that the Type A-MSN and Type B-MSN are both derived from LCLC1, whereas Liseo and Loperfido [14] construction is derived from LCLC2.

3.4. *Gupta, González-Farías and Domínguez-Molina class of MSN*

Domínguez-Molina et al. [8] constructed a more general class (Generalized Multivariate Skew Normal, or GMSN) which includes Type B-MSN as a special case. The density is of the form

$$f_{p,q}(x; \mu, \Sigma, D, v, \Delta) = \frac{\Phi_q(Dx; v, \Delta)}{\Phi_q(D\mu; v, \Delta + D\Sigma D')}\phi_p(x; \mu, \Sigma). \tag{6}$$

They used notation $X \sim SN_{p,q}(\mu, \Sigma, D, v, \Delta)$. If we take $v = D\mu$ and $\Delta = I_p$ then the density reduces to

$$f_{p,q}(x; \mu, \Sigma, D, D\mu, I_p) = \frac{\Phi_p\left(D(x-\mu); 0, I_p\right)}{\Phi_p(0; 0, I + D\Sigma D')}\phi_p(x; \mu, \Sigma) \\ \sim \text{Type B-MSN}(D, \mu, \Omega).$$

Note that, Domínguez-Molina et al. [8] is a reparameterization of LCLC1. To obtain their results, we only take normal distribution in our elliptical distribution framework and define

$$\begin{cases} \begin{pmatrix} Z \\ X \end{pmatrix} \sim MN\left(\begin{pmatrix} v \\ \mu \end{pmatrix}, \begin{pmatrix} \Delta & 0 \\ 0 & \Sigma \end{pmatrix}\right) \\ \begin{pmatrix} I & -D \end{pmatrix}\begin{pmatrix} Z \\ X \end{pmatrix} \leqslant 0 \end{cases}$$

and calculate the marginal density of X, i.e., $R_1 = I_{q\times q}$, $R_2 = -D_{q\times p}$, $\Omega = \begin{pmatrix} \Delta & 0 \\ 0 & \Sigma \end{pmatrix}$. The calculation of the density of X given $Z - DX \leqslant 0$ produces Domínguez-Molina et al. [8] results.

On the other hand, for normal case, density (4) can be written as GMSN with parameters $\mu = C_2\mu_2$, $\Sigma = C_2\Omega_{22}C_2'$, $D = -(R_2 + R_1\Omega_{12}\Omega_{22}^{-1})C_2^{-1}$, $v = R_1\mu_1 + d - R_1\Omega_{12}\Omega_{22}^{-1}\mu_2$, $\Delta = \Omega_x = R_1(\Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21})R_1'$. It is easy to check that $\Delta + D\Sigma D' = R\Omega R'$. Similarly, density (5) can be written as GMSN with parameters $\mu = \mu_2^{wx}$, $\Sigma = \Omega_{22}^{wx}$, $D = -(R_2C_2^{-1} + C_3\Omega_{12}^{wx}\Omega_{22}^{wx-1})$, $v = C_3(\mu_1^{wx} - \Omega_{12}^{wx}\Omega_{22}^{wx-1}\mu_2^{wx}) + d$, $\Delta = C_3\Omega_{11}^{*}C_3'$, where

$$\begin{cases} C_3 &= R_1C_1^{-1} - R_2C_2^{-1}, \\ \mu_2^{wx} &= \begin{pmatrix} C_1\mu_1 \\ C_1\mu_1 + C_2\mu_2 \end{pmatrix}, \\ \Omega^{wx} &= \begin{pmatrix} \Omega_{11}^{wx} & \Omega_{12}^{wx} \\ \Omega_{21}^{wx} & \Omega_{22}^{wx} \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ C_1 & C_2 \end{pmatrix}\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}\begin{pmatrix} C_1' & C_1' \\ 0 & C_2' \end{pmatrix}, \\ &= \begin{pmatrix} C_1\Omega_{11}C_1' & C_1\left(\Omega_{11}C_1' + \Omega_{12}C_2'\right) \\ \left(C_1\Omega_{11} + C_2\Omega_{21}\right)C_1' & \left(C_1\Omega_{11} + C_2\Omega_{21}\right)C_1' + \left(C_1\Omega_{12} + C_2\Omega_{22}\right)C_2' \end{pmatrix}, \\ \Omega_{11}^{*} &= \Omega_{11}^{wx} - \Omega_{12}^{wx}\Omega_{22}^{wx-1}\Omega_{21}^{wx}, \text{ and} \\ a^{*} &= (x - \mu_2^{wx})'\Omega_{22}^{wx-1}(x - \mu_2^{wx}). \end{cases}$$

3.5. Skew elliptical distribution

Here we consider two types of skew elliptical distributions and demonstrate how they can be obtained from our general formulation.

3.5.1. Skew elliptical distribution $SE_k(\mu, \Omega, \delta; g^{(k+1)})$

Branco and Dey [7] consider $Y^* = (Y_0, Y_1, Y_2, \ldots, Y_k)' \sim El_{k+1}(\mu^*, \Sigma; \phi)$, where $\mu^* = (0, \mu)$, $\mu = (\mu_1, \ldots, \mu_k)'$, ϕ is the characteristic function with the scale parameter matrix Σ having the form

$$\Sigma = \begin{pmatrix} 1 & \delta' \\ \delta & \Omega \end{pmatrix}$$

with $\delta = (\delta_1, \ldots, \delta_k)'$. Let $X = [Y|Y_0 > 0]$, where $Y = (Y_1, Y_2, \ldots, Y_k)'$, then

$$f_X(x; \mu, \Omega, \delta; g^{(k+1)}) = 2 f_{g^{(k)}} F_{g_{q(x)}}(\lambda'(x-\mu)), x \in \Re^p,$$

where $f_{g^{(k)}}$ is the pdf of an elliptical distribution with generator function $g^{(k)}(\bullet)$ and $F_{g_{q(x)}}$ is the cdf of a univariate elliptical distribution with $g_{q(x)}$ as the generator function. It includes Type A-MSN (α, μ, Ω) as a special case. It is a special case of LCLC1 with $R_1 = -1$, $R_2 = (0, 0, \ldots, 0)'$, $Y_1 = Y_0$, $Y_2 = (Y_1, Y_2, \ldots, Y_k)'$, $d = 0$ and $C_2 = I_{k\times k}$.

3.5.2. *Skew elliptical distribution* $SE(\mu, \Sigma, D; g^{(m)})$

Sahu et al. [17] use the following construction. Suppose ε and Z are two m-dimensional random vectors. Let μ be an m-dimensional vector and Σ be an $m \times m$ positive definite matrix. Assume that

$$Y = \begin{pmatrix} \varepsilon \\ Z \end{pmatrix} \sim El\left(\begin{pmatrix} \mu \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & 0 \\ 0 & I \end{pmatrix}; g^{(2m)} \right),$$

where 0 is the null matrix and I is the identity matrix. They consider a skew elliptical class of distributions by using the transformation

$$X = DZ + \varepsilon,$$

where D is a diagonal matrix with elements $\delta_1, \ldots, \delta_m$. Let $\delta' = (\delta_1, \ldots, \delta_m)$. The class is developed by considering the random variable $[X|Z > 0]$ where $Z > 0$ means $Z_i > 0$ for $i = 1, \ldots, m$. Then

$$\begin{aligned} f_X(x; \mu, \Sigma, D; g^{(m)}) \;=\; & 2^m f_X(x|\mu, \Sigma + D^2; g^{(m)}) \\ & \times F((I - D(\Sigma + D^2)^{-1} D)^{-\frac{1}{2}} D(\Sigma + D^2)^{-1}(x-\mu)|0, I; g^{(m)}_{q(x-\mu)}), \end{aligned}$$

where

$$g_a^{(m)}(u) = \frac{\Gamma(m/2)}{\pi^{m/2}} \frac{g(a+u; 2m)}{\int_0^\infty r^{m/2-1} g(a+r; 2m)\, dr}, a > 0$$

and

$$q(x-\mu) = (x-\mu)'(\Sigma + D^2)^{-1}(x-\mu).$$

This density matches with the one obtained by Branco and Dey [7] only in the univariate case. The derived skew normal distribution is

$$\begin{aligned} f(x|\mu, \Sigma, D) \;=\; & 2^m |\Sigma + D^2|^{-\frac{1}{2}} \phi_m((\Sigma + D^2)^{-\frac{1}{2}}(x-\mu)) \\ & \times \Phi_m((I - D(\Sigma + D^2)^{-1} D)^{-\frac{1}{2}} D(\Sigma + D^2)^{-1}(x-\mu)), \end{aligned}$$

where ϕ_m and Φ_m denote, respectively, the density and cdf of an m-dimensional normal distribution with mean 0 and covariance matrix identity. Clearly this is different from

Type A-MSN distribution. Note that this is a special case of LCLC2 with

$$\mu = \begin{pmatrix} 0_{m\times 1} \\ \mu_{m\times 1} \end{pmatrix}, \Omega = \begin{pmatrix} I_{m\times m} & 0 \\ 0 & \Sigma_{m\times m} \end{pmatrix}, R_1 = -I_{m\times m}, R_2 = 0, d = 0_{m\times 1},$$

$$C_1 = D_{m\times m} \text{ and } C_2 = I_{m\times m}.$$

Remark 3.1. We give a straightforward demonstration by LCLC1 parameterization for the closure of the marginal distribution in Domínguez-Molina et al. [8] class.

In (6), $X \sim SN_{p,q}(\mu, \Sigma, D, v, \Delta)$ is partitioned into two components, X_1 and X_2, of dimensions k and p–k space, respectively. Then the marginal distribution of X_1 is

$$SN_{k,q}(\mu_1, \Sigma_{11}, D_1 + D_2\Sigma_{21}\Sigma_{11}^{-1}, v + D_2(\Sigma_{21}\Sigma_{11}^{-1}\mu_1 - \mu_2), \Delta \\ + D_2(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})D_2'),$$

where

$$\mu = \begin{pmatrix} \mu_{1_{k\times 1}} \\ \mu_{2_{(p-k)\times 1}} \end{pmatrix}, v = \begin{pmatrix} v_{1_{k\times 1}} \\ v_{2_{(p-k)\times 1}} \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11_{k\times k}} & \Sigma_{12_{k\times(p-k)}} \\ \Sigma_{21_{(p-k)\times k}} & \Sigma_{22_{(p-k)\times(p-k)}} \end{pmatrix}$$

and

$$D = \begin{pmatrix} D_{1_{q\times k}} & D_{2_{q\times(p-k)}} \end{pmatrix}.$$

Domínguez-Molina et al. [8] use moment generating function and lengthy algebraic derivation (Appendix A) to get the marginal density. Here we use our LCLC1 set up to obtain the result in a straightforward way.

Let $X_1 = Z$, $X_2 = (A_1', A_2')'$,

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} Z \\ A_2 - \Sigma_{21}\Sigma_{11}^{-1}A_1 \\ A_1 \end{pmatrix}.$$

The constraint $Z - DA \leqslant 0$ is equivalent to $Y_1 - D_2Y_2 - (D_1 + D_2\Sigma_{21}\Sigma_{11}^{-1})Y_3 \leqslant 0$. Further it is easy to see that $(Y_1 - D_2Y_2, (D_1 + D_2\Sigma_{21}\Sigma_{11}^{-1})Y_3)$ are independent. Now

$$\begin{cases} Var(Y_1 - D_2Y_2) = \Delta + D_2\Sigma_{22.1}D_2', \\ Var(Y_3) = \Sigma_{11}, \\ D = D_1 + D_2\Sigma_{21}\Sigma_{11}^{-1}, \\ v = v + D_2(\Sigma_{21}\Sigma_{11}^{-1}\mu_1 - \mu_2), \quad \text{and} \\ \mu = \mu_1. \end{cases}$$

Thus the marginal density for A_1 is

$$SN_{k,q}(\mu_1, \Sigma_{11}, D_1 + D_2\Sigma_{21}\Sigma_{11}^{-1}, v + D_2(\Sigma_{21}\Sigma_{11}^{-1}\mu_1 - \mu_2), \Delta + D_2\Sigma_{22.1}D_2').$$

Remark 3.2. We give a straightforward proof by LCLC1 parameterization for the closure of conditional distribution in Domínguez-Molina et al. [8] class.

Suppose $X \sim SN_{p,q}(\mu, \Sigma, D, v, \Delta)$ and X is partitioned in two components, X_1 and X_2, of dimensions k and $p-k$, respectively. Then the conditional distribution of X_1 given X_2 is

$$SN_{k,q}(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}, D_1, v - D_2x_2, \Delta),$$

where

$$\mu = \begin{pmatrix} \mu_{1_{k\times 1}} \\ \mu_{2_{(p-k)\times 1}} \end{pmatrix}, v = \begin{pmatrix} v_{1_{k\times 1}} \\ v_{2_{(p-k)\times 1}} \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11_{k\times k}} & \Sigma_{12_{k\times(p-k)}} \\ \Sigma_{21_{(p-k)\times k}} & \Sigma_{22_{(p-k)\times(p-k)}} \end{pmatrix}$$

and

$$D = \begin{pmatrix} D_{1_{q\times k}} & D_{2_{q\times(p-k)}} \end{pmatrix}.$$

Domínguez-Molina et al. [8] use lengthy algebraic derivations to get the conditional density. Clearly using our LCLC1 set up we obtain the result in a straightforward way.

Let the original random vector be partitioned as $(Z', X_1', X_2')'$, and

$$\begin{pmatrix} Z \\ X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} v \\ \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Delta & 0 \\ 0 & \Sigma \end{pmatrix}\right)$$

then

$$\begin{pmatrix} Z \\ X_1 \end{pmatrix} | X_2 = x_2 \sim N\left(\begin{pmatrix} v \\ \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2) \end{pmatrix}, \begin{pmatrix} \Delta & 0 \\ 0 & \Sigma_{11.2} \end{pmatrix}\right).$$

Conditionally, Z and X_1 given X_2 are still independent. Under the constraint $Z - DA = Z - (D_1, D_2)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \leqslant 0$ if and only if $(Z - D_2X_2) - D_1X_1 \leqslant 0$. Now given X_2

$$\begin{cases} \Delta &= Var(Z - D_2X_2 | X_2 = x_2), \\ \Sigma &= Var(X_1 | X_2 = x_2) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}, \\ D &= D_1, \\ v &= v - D_2X_2, \quad \text{and} \\ \mu &= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2). \end{cases}$$

Thus we get the conditional density for $X_1 | X_2 = x_2$ under the linear constraint.

4. Moment generating functions

First, we develop moment generating function of both LCLC1 and LCLC2 class of skew normal distribution.

Lemma 4.1. *Let U be a k-dimensional vector and let B be a $k \times p$ matrix. If $V \sim N_p(\mu_1, \Sigma)$, then*

$$E_V\left[\Phi_k(u + BV; \mu_2, \Omega)\right] = \Phi_k(u - \mu_2 + B\mu_1; 0, B\Sigma B' + \Omega).$$

Proof. The proof is given in Box and Tiao [6]. □

Lemma 4.2.

$$(x-\mu)'\Sigma^{-1}(x-\mu)-2t'x=(x-\mu-\Sigma t)'\Sigma^{-1}(x-\mu-\Sigma t)-2\mu' t-t'\Sigma t.$$

Proof. The proof follows from direct calculation. □

Theorem 4.1. *Under LCLC1 condition, the moment generating function of X is*

$$\begin{aligned}m_x(t) = {} & \frac{1}{\Phi(-d-R\mu|0,R\Omega R')}\exp(\mu_2'C_2't+\frac{1}{2}t'C_2\Omega_{22}C_2't)\\ & \times\Phi_p(-d-R_1(\mu_1-\Omega_{12}\Omega_{22}^{-1}\mu_2)\\ & -(R_2-R_1\Omega_{12}\Omega_{22}^{-1})(\mu_2+\Omega_{22}C_2't);0,(R_2-R_1\Omega_{12}\Omega_{22}^{-1})\\ & \times\Omega_{22}(R_2-R_1\Omega_{12}\Omega_{22}^{-1})'+R_1(\Omega_{11}-\Omega_{12}\Omega_{22}^{-1}\Omega_{21})R_1').\end{aligned}$$

Proof. The proof follows from direct calculation using the pdf of X having form (3). □

Theorem 4.2. *Under LCLC2 condition, the moment generating function of X is*

$$\begin{aligned}m_x(t) = {} & \frac{1}{\Phi(-d-R\mu|0,R\Omega R')}\exp(\mu_2^{wx\prime}t+\frac{1}{2}t'\Omega_{22}^{wx}t)\\ & \times\Phi_p(-d-C_3(\mu_1^{wx}-\Omega_{12}^{wx}\Omega_{22}^{wx-1}\mu_2^{wx})\\ & -(R_2C_2^{-1}-C_3\Omega_{12}^{wx}\Omega_{22}^{wx-1})(\mu_2^{wx}+\Omega_{22}^{wx}t);0,(R_2C_2^{-1}-C_3\Omega_{12}^{wx}\Omega_{22}^{wx-1})\\ & \times\Omega_{22}^{wx}(R_2C_2^{-1}-C_3\Omega_{12}^{wx}\Omega_{22}^{wx-1})'\\ & +C_3(\Omega_{11}^{wx}-\Omega_{12}^{wx}\Omega_{22}^{wx-1}\Omega_{21}^{wx})C_3').\end{aligned}$$

Proof. The proof follows from direct calculation using the pdf of X having form (4). □

Remark 4.1. In LCLC1 setup, let $X_a \sim GSN(\mu_a, \Omega_a, R_a, d_a, C_{xa})$, $X_b \sim GSN(\mu_b, \Omega_b, R_b, d_b, C_{xb})$. Further suppose X_a is independent of X_b. Then the moment generating function of the joint distribution of (X_a', X_b') is obtained by computing

$$\begin{aligned}M_{X_a,X_b}(t) &= M_{X_a,X_b}(t_a,t_b)=Ee^{t_a'X_a+t_b'X_b}=M_{X_a}(t_a)M_{X_b}(t_b)\\ &= \frac{1}{\Phi(-d-R\mu|0,R\Omega R')}\exp(\mu_2'C_x't+\frac{1}{2}t'C_x\Omega_{22}C_x't)\\ &\quad\times\Phi_p(-d-R_1(\mu_1-\Omega_{12}\Omega_{22}^{-1}\mu_2)-(R_2-R_1\Omega_{12}\Omega_{22}^{-1})\\ &\quad\times(\mu_2+\Omega_{22}C_x't);0,(R_2-R_1\Omega_{12}\Omega_{22}^{-1})\Omega_{22}(R_2-R_1\Omega_{12}\Omega_{22}^{-1})'\\ &\quad+R_1(\Omega_{11}-\Omega_{12}\Omega_{22}^{-1}\Omega_{21})R_1'),\end{aligned}$$

where

$$\mu=\begin{pmatrix}\mu_a\\ \mu_b\end{pmatrix},\Omega=\begin{pmatrix}\Omega_a & 0\\ 0 & \Omega_b\end{pmatrix},R=\begin{pmatrix}R_a & 0\\ 0 & R_b\end{pmatrix},d=\begin{pmatrix}d_a\\ d_b\end{pmatrix},$$

$$C_x = \begin{pmatrix} C_{xa} & 0 \\ 0 & C_{xb} \end{pmatrix}, t = \begin{pmatrix} t_a \\ t_b \end{pmatrix}.$$

Hence the joint distribution of $(X_1, X_2)'$ is $GSN(\mu, \Omega, R, d, C_x)$.

Remark 4.2. In LCLC2 setup, let $X_a \sim GSN(\mu_a, \Omega_a, R_a, d_a, C_{a1}, C_{a2})$, and $X_b \sim GSN(\mu_b, \Omega_b, R_b, d_b, C_{b1}, C_{b2})$. Further suppose X_a is independent of X_b. Then the moment generating function of the joint distribution of (X_a', X_b') is obtained by computing

$$\begin{aligned} M_{X_a,X_b}(t) &= M_{X_a,X_b}(t_a, t_b) = Ee^{t_a' X_a + t_b' X_b} = M_{X_a}(t_a) M_{X_b}(t_b) \\ &= \frac{1}{\Phi(-d - R\mu|0, R\Omega R')} \exp(\mu_2^{wx\prime} t + \frac{1}{2} t' \Omega_{22}^{wx} t) \\ &\quad \times \Phi_p(-d - C_3(\mu_1^{wx} - \Omega_{12}^{wx} \Omega_{22}^{wx-1} \mu_2^{wx}) \\ &\quad -(R_2 C_2^{-1} - C_3 \Omega_{12}^{wx} \Omega_{22}^{wx-1}) \\ &\quad \times(\mu_2^{wx} + \Omega_{22}^{wx} t); 0, (R_2 C_2^{-1} - C_3 \Omega_{12}^{wx} \Omega_{22}^{wx-1}) \\ &\quad \times \Omega_{22}^{wx} (R_2 C_2^{-1} - C_3 \Omega_{12}^{wx} \Omega_{22}^{wx-1})' \\ &\quad + C_3(\Omega_{11}^{wx} - \Omega_{12}^{wx} \Omega_{22}^{wx-1} \Omega_{21}^{wx}) C_3'), \end{aligned}$$

where

$$\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}, \Omega = \begin{pmatrix} \Omega_a & 0 \\ 0 & \Omega_b \end{pmatrix}, R = \begin{pmatrix} R_a & 0 \\ 0 & R_b \end{pmatrix}, d = \begin{pmatrix} d_a \\ d_b \end{pmatrix},$$

$$C_1 = \begin{pmatrix} C_{a1} & 0 \\ 0 & C_{b1} \end{pmatrix}, C_2 = \begin{pmatrix} C_{a2} & 0 \\ 0 & C_{b2} \end{pmatrix}, t = \begin{pmatrix} t_a \\ t_b \end{pmatrix},$$

hence the joint distribution of $(X_1, X_2)'$ is $GSN(\mu, \Omega, R, d, C_1, C_2)$.

Remark 4.3. González-Farías et al. [9,10] formalized closed skew normal (CSN) family (6) and showed it is closed under full row rank linear transformations and full column rank linear transformations (defining the singular skew normal distribution). The closure property also applies for LCLC1/LCLC2 construction under reparameterization (Section 3.4).

Remark 4.4. In LCLC1 setup, suppose $X \sim GSN(\mu, \Omega, R, d, C_x)$, and A is a non-singular square matrix, then $AX \sim GSN(\mu, \Omega, R, d, AC_x)$, which means all the parameters are the same except C is replaced by AC. Thus any permutation of $X = (X_1, X_2, \ldots, X_p)'$ is closed within LCLC1 setup.

Remark 4.5. In LCLC2 setup, suppose $X \sim GSN(\mu, \Omega, R, d, C_1, C_2)$, and A is a non-singular square matrix, then

$$AX \sim GSN(\mu, \Omega, R, d, AC_1, AC_2)$$

which means all the parameters are the same except C_1 and C_2 are replaced by AC_1 and AC_2, respectively.

Remark 4.6. In LCLC1 setup, suppose $X \sim GSN(\mu, \Omega, R, d, C_x)$, and A is a non-singular square matrix, then

$$AX + b \sim GSN(\mu', \Omega', R', d', C_x'),$$

where

$$\begin{cases} \mu' = \begin{pmatrix} \mu_1' \\ \mu_2' \end{pmatrix} = \begin{pmatrix} \mu_1 \\ AC_x\mu_2 + b \end{pmatrix}, \\ \Omega' = \begin{pmatrix} \Omega_{11}' & \Omega_{12}' \\ \Omega_{21}' & \Omega_{22}' \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12}C'A' \\ AC\Omega_{21} & AC\Omega_{22}C'A' \end{pmatrix}, \\ R' = \begin{pmatrix} R_1' \\ R_2' \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2C_x^{-1}A^{-1} \end{pmatrix}, \\ d' = d - R_2C_x^{-1}A^{-1}b, \text{ and} \\ C_x' = I. \end{cases}$$

It is clear to see that the representation is not unique based on the five parameters.

Remark 4.7. In LCLC2 setup, if $X \sim GSN(\mu, \Omega, R, d, C_1, C_2)$, and A is a non-singular square matrix, then

$$AX + b \sim GSN(\mu', \Omega', R', d', C_1', C_2'),$$

where

$$\begin{cases} \mu' = \begin{pmatrix} \mu_1' \\ \mu_2' \end{pmatrix} = \begin{pmatrix} \mu_1 + p(AC_1)^{-1}b \\ \mu_2 + q(AC_2)^{-1}b \end{pmatrix}, \\ \Omega' = \begin{pmatrix} \Omega_{11}' & \Omega_{12}' \\ \Omega_{21}' & \Omega_{22}' \end{pmatrix} = \Omega, \\ R' = \begin{pmatrix} R_1' \\ R_2' \end{pmatrix} = R, \\ d' = d - pR_1(AC_1)^{-1}b - qR_2(AC_2)^{-1}b, \quad \text{and} \\ C_1' = AC_1, C_2' = AC_2. \end{cases}$$

The representation is again not unique based on the five parameters at least for any $p + q = 1$.
Now the following two theorems provide distribution of quadratic form under LCLC1 and LCLC2 setup through the moment generating functions.

Theorem 4.3. *In LCLC1 setup, suppose $X \sim GSN(\mu, \Omega, R, d, C_x)$ and A is an symmetric idempotent matrix, then the mgf of $X'AX$ is given as*

$$M_{X'AX}(t) = \frac{1}{\Phi_p(-d - R\mu; 0, R\Omega R')} \times |I - 2tAC_2\Omega_{22}C_2'|^{-1/2}$$

$$
\begin{aligned}
&\times \exp\left(-\frac{1}{2}\mu_2'\Omega_{22}^{-1}C_2^{-1}\left(I-(I-2tAC_2\Omega_{22}C_2')^{-1}\right)C_2\mu_2\right)\\
&\times\Phi_p(-d-R_1(\mu_1-\Omega_{12}\Omega_{22}^{-1}\mu_2)-(R_2+R_1\Omega_{12}\Omega_{22}^{-1})\\
&\times C_2^{-1}\mu_2'C_2'(I-2tAC_2\Omega_{22}C_2')^{-1};0,(R_2+R_1\Omega_{12}\Omega_{22}^{-1})\\
&\times\Omega_{22}C_2'(I-2tAC_2\Omega_{22}C_2')^{-1}C_2'^{-1}\\
&\times(R_2+R_1\Omega_{12}\Omega_{22}^{-1})+\Omega_x).
\end{aligned}
$$

Proof. Recall the identity in Searle [18],

$$
\begin{aligned}
\exp(tx'Ax)\phi_p(x;\mu,\Sigma) = {}&|I-2tA\Sigma|^{-1/2}\exp\left(-\frac{1}{2}\mu'\Sigma^{-1}(I-(I-2tA\Sigma)^{-1})\right)\\
&\times\phi_p(x;\mu'(I-2tA\Sigma)^{-1},\Sigma(I-2tA\Sigma)^{-1})
\end{aligned}
$$

and recall the LCLC1 density function in Theorem 2.2.1 and identity in Lemma 4.1. The result follows after some simplifications.

Theorem 4.4. *In LCLC2 setup, suppose $X \sim GSN(\mu,\Omega,R,d,C_1,C_2)$, and A is an symmetric idempotent matrix, then the mgf of $X'AX$ is*

$$
\begin{aligned}
M_{X'AX}(t) = {}&\frac{1}{\Phi_p(-d-R\mu;0,R\Omega R')}\\
&\times|I-2tA\Omega_{22}^{wx}|^{-1/2}\exp\left(-\frac{1}{2}\mu_2^{wx\prime}\Omega_{22}^{wx\,-1}(I-(I-2tA\Omega_{22}^{wx})^{-1})\right)\\
&\times\Phi_p(-d-C_3(\mu_1^{wx}-\Omega_{12}^{wx}\Omega_{22}^{wx\,-1}\mu_2^{wx})\\
&-(R_2C_2^{-1}+C_3\Omega_{12}^{wx}\Omega_{22}^{wx\,-1})(I-2tA\Omega_{22}^{wx})'^{-1}\mu_2^{wx};0,(R_2C_2^{-1}\\
&+C_3\Omega_{12}^{wx}\Omega_{22}^{wx\,-1})\Omega_{22}^{wx}(I-2tA\Omega_{22}^{wx})^{-1}(R_2C_2^{-1}\\
&+C_3\Omega_{12}^{wx}\Omega_{22}^{wx\,-1})'+C_3\Omega_{11}^{*}C_3').
\end{aligned}
$$

Proof. The proof follows using similar idea used in the previous proof. □

For scale mixture of normal class of general elliptical distributions such as multivariate t-distribution, the moment generating function of both X and quadratic forms can be represented as an integral with respect to a mixing distribution. Further details are given in Liu and Dey [15].

5. Concluding remarks

The new class of skewed distributions obtained in this article is very general, quite flexible and widely applicable. Even for $\min(X,Y)$ case in Roberts [16], where (X,Y) was correlated bivariate normal random variable, it can easily be seen to be a mixture of two univariate skew normal densities on two regions $X=\min(X,Y)$ on $(X\leqslant Y)$ and $Y=\min(X,Y)$ on $(Y\leqslant X)$. Thus our construction based on linear combination with linear constraint is very general one and includes different types of skew distributions exist in the literature. Although the associated density functions are quite difficult to handle we can show that many special cases such models can be easily fit using MCMC methods. This paper creates a theoretical foundation on general skew elliptical distribution and is anticipated that further theoretical development relating to the distribution of ratio of quadratic forms, robustness study of Student's *t*-test and *F*-test can be obtained under this general skew elliptical distributions.

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